

ON THE REAL RANK OF MONOMIALS

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ABSTRACT. In this paper we study the real rank of monomials and we give an upper bound for the real rank of all monomials. We show that the real and the complex ranks of a monomial coincide if and only if the least exponent is equal to one.

1. INTRODUCTION

Let \mathbb{K} be a field and $S = \mathbb{K}[x_0, \dots, x_n] = \bigoplus_{d \geq 0} S_d$ be the ring of polynomials with coefficients in \mathbb{K} and with standard grading, i.e. S_d is the \mathbb{K} -vector space of homogeneous polynomials, or forms, of degree d .

Given $F \in S_d$, we define a *Waring decomposition of F over \mathbb{K}* as a sum

$$(1) \quad F = \sum_{i=1}^s c_i L_i^d,$$

where $c_i \in \mathbb{K}$ and the L_i 's are linear forms over \mathbb{K} . The smallest s for which such a decomposition exists is called *Waring rank of F over \mathbb{K}* and it is denoted by $\text{rk}_{\mathbb{K}}(F)$.

The study of Waring decompositions over the complex numbers goes back to the work of Sylvester [16] and other geometers and algebraists of the XIX century; see [11] for historical details. Even if it has a long history, it was only in 1995 that the Waring ranks were determined for *general* forms over the complex numbers; see [1].

However, the computation of the Waring rank is not known for all forms. The case of complex binary forms goes back to Sylvester and has been recently reviewed in [8]. More recently, some progress has been made: the complex Waring rank of monomials (and sums of pairwise coprime monomials) and complex ranks of other sporadic families of polynomials have been determined in [7] and [6] respectively. Some algorithms have been proposed, but they require technical restrictions to compute the rank; see [2, 4, 9, 13].

The computation of Waring ranks over the real numbers is even more difficult. For instance, the real rank for monomials is only known in the case of two variables; see [3]. In [15], results on the rank of binary forms over the reals and other fields are exhibited.

In the present paper we study the connection between the complex and real rank of monomials.

The paper is structured as follows. In Section 2, we introduce notation and background. In Section 3, we prove our main results. We give an upper bound for the real rank of monomials in Theorem 3.1. In Proposition 3.6 we show that this upper bound is not

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always sharp. In Theorem 3.5, we prove that for monomials the real and complex rank coincide if and only if their least exponent is equal to one.

2. BACKGROUND

Let $T = \mathbb{K}[X_0, \dots, X_n] = \bigoplus_{d \geq 0} T_d$ be the dual ring of S acting by differentiation on S :

$$X_i \circ x_j := \frac{\partial}{\partial x_i} x_j.$$

For any homogeneous polynomial $F \in S$, the *perp ideal* of F is

$$F^\perp := \{\partial \in T \mid \partial F = 0\} \subset T.$$

One of the key results to study the Waring problem is the Apolarity Lemma.

Lemma 2.1 (APOLARITY LEMMA, [11, Lemma 1.15]). *Let $F \in S$ be a form of degree d . The following are equivalent:*

- (1) $F = \sum_{i=1}^r L_i^d$, where the L_i 's are linear forms;
- (2) $I_{\mathbb{X}} \subset F^\perp$, where $I_{\mathbb{X}}$ is the ideal defining a set \mathbb{X} of r reduced points.

A set of reduced points \mathbb{X} in \mathbb{P}^n is said to be *apolar* to F if $I_{\mathbb{X}} \subset F^\perp$.

As already mentioned, the complex rank of monomials has been determined in [7].

Theorem 2.2 ([7, Corollary 3.3]). *Let $M = x_0^{a_0} x_1^{a_1} \dots x_n^{a_n}$ with $0 < a_0 \leq a_1 \leq \dots \leq a_n$. Then*

$$\mathrm{rk}_{\mathbb{C}}(M) = \frac{1}{a_0 + 1} \prod_{i=0}^n (a_i + 1).$$

Moreover, any set of reduced points apolar to a monomial M , whose cardinality is equal to $\mathrm{rk}_{\mathbb{C}}(M)$, is a complete intersection.

Theorem 2.3 ([5, Theorem 1]). *Let $M = x_0^{a_0} x_1^{a_1} \dots x_n^{a_n}$ with $0 < a_0 \leq a_1 \leq \dots \leq a_n$. Then, any ideal $I_{\mathbb{X}}$ of a set of $\mathrm{rk}_{\mathbb{C}}(M)$ points apolar to M is a complete intersection of the form*

$$(X_1^{a_1+1} - F_1 X_0^{a_0+1}, \dots, X_n^{a_n+1} - F_n X_0^{a_0+1}),$$

where the forms F_i 's have degrees $a_i - a_0$.

The real rank of monomials in two variables has been computed in [3].

Proposition 2.4 ([3, Proposition 4.4]). *If $M = x_0^{a_0} x_1^{a_1}$, then $\mathrm{rk}_{\mathbb{R}}(M) = a_0 + a_1$.*

In view of these results, we can easily note that whenever one of the exponents of a binary monomial is equal to one, then real and complex rank coincide. Hence, in the case of two variables the monomials whose real and complex rank coincide are those whose least exponent is one. The proof of Proposition 2.4 is a combination of Apolarity Lemma with a straightforward application of the Descartes' rule of signs.

Lemma 2.5 ([3, Lemma 4.1– 4.2]). *Let F be a real polynomial in one variable*

$$F = x^d + c_1 x^{d-1} + \dots + c_{d-1} x + c_d.$$

Then

- (1) *for any $i < d$, there is a choice of c_j 's such that F has distinct real roots and $c_i = 0$;*
- (2) *if $c_i = c_{i+1} = 0$ for some $0 < i < d$, then F has a non real root.*

In order to prove the characterization of monomials whose complex and real ranks are equal, featured in Theorem 3.5, we recall the trace bilinear form of finite \mathbb{K} -algebras; we refer to [14] for details.

Let $A = \mathbb{K}[X_0, \dots, X_n]/I$ be a finite \mathbb{K} -algebra. For any element $F \in A$, we define the endomorphism $m_F \in \text{End}(A)$ to be the multiplication by F . Since A is a finite dimensional \mathbb{K} -vector space, we have a trace map $\text{Tr}_{A/\mathbb{K}} : \text{End}(A) \rightarrow \mathbb{K}$, which is the trace of the corresponding matrix. We define a symmetric bilinear form

$$B(F, G) : A \otimes A \rightarrow \mathbb{K},$$

by

$$B(F, G) = \text{Tr}(m_F \cdot m_G) = \text{Tr}_{A/\mathbb{K}}(m_{F \cdot G}).$$

The following result is featured in [14, Thm. 2.1]; we give an elementary proof for the sake of completeness.

Proposition 2.6. *Let A be a reduced finite \mathbb{R} -algebra of dimension N . If $\text{Spec} A$ consists only of \mathbb{R} -points, then the bilinear form*

$$B : A \otimes A \rightarrow \mathbb{R}, (F, G) \mapsto \text{Tr}_{A/\mathbb{R}}(m_{F \cdot G})$$

is positive definite.

Proof. The \mathbb{R} -algebra A is isomorphic to $\mathbb{R} \times \dots \times \mathbb{R}$ because A is reduced. The representing matrix of the \mathbb{R} -linear map $A \rightarrow A$ given by multiplication with an element $F = (F_1, \dots, F_N) \in A$ is the diagonal matrix with diagonal entries F_1, \dots, F_N . Thus, we have $B(F, F) = \text{Tr}_{A/\mathbb{R}}(m_{F^2}) = F_1^2 + \dots + F_N^2 \geq 0$ and $B(F, F) = 0$ if and only if $F = 0$. Hence B is positive definite. \square

3. REAL AND COMPLEX RANKS OF MONOMIALS

In this section we prove our main results. First, we give an upper bound for the real rank of monomials.

Theorem 3.1. *If $M = x_0^{a_0} \dots x_n^{a_n}$ with $0 < a_0 \leq \dots \leq a_n$, then*

$$\text{rk}_{\mathbb{R}}(M) \leq \frac{1}{2a_0} \prod_{i=0}^n (a_i + a_0).$$

Proof. Clearly $M^\perp = (X_0^{a_0+1}, \dots, X_n^{a_n+1})$. Let us consider

$$G_i = X_0^{a_0+1}g_i(X_0, X_i) + X_i^{a_i+1}h_i(X_0, X_i),$$

where $\deg g_i = a_i - 1$ and $\deg h_i = a_0 - 1$, for every $i = 1, \dots, n$.

Each G_i is a binary form of degree $a_0 + a_i$ where the monomial $X_0^{a_0}X_i^{a_i}$ does not appear. Thus, by Lemma 2.5, there exists a choice of g_i and h_i such that G_i has $a_0 + a_i$ distinct real roots, say $p_{i,j}$ with $j = 1, \dots, a_0 + a_i$. Therefore, the ideal $(G_1, \dots, G_n) \subset M^\perp$ is the ideal of the following set of distinct real points:

$$\mathbb{X} = \left\{ [1 : p_{1,j_1} : \dots : p_{n,j_n}] \mid 1 \leq j_i \leq a_0 + a_i, \text{ for } i = 1, \dots, n \right\}.$$

By the Apolarity Lemma, the proof is complete. \square

As a direct corollary, we have that whenever the least exponent of a monomial is one, then the real and the complex rank coincide.

Corollary 3.2. *If $M = x_0x_1^{a_1} \dots x_n^{a_n}$, then $\text{rk}_{\mathbb{C}}(M) = \text{rk}_{\mathbb{R}}(M)$.*

Proof. If $a_0 = 1$, the upper bound given by Theorem 3.1 equals the complex rank of M given by the formula in Theorem 2.2. \square

Remark 3.3. Here we produce a family of minimal real Waring decompositions for $M = x_0x_1^{a_1} \dots x_n^{a_n}$. Consider a set of real numbers

$$\left\{ p_{i,j} \in \mathbb{R} \mid i = 1, \dots, n \text{ and } j = 1, \dots, a_i + 1 \right\},$$

such that

$$\sum_{j=1}^{a_i+1} p_{i,j} = 0, \text{ for any } i,$$

and $p_{i,a} \neq p_{i,b}$ if $a \neq b$, for any i . Hence, we have a set \mathbb{X} of $(a_1 + 1) \cdot \dots \cdot (a_n + 1)$ distinct reduced real points in \mathbb{P}^n given by

$$\mathbb{X} = \left\{ [1 : p_{1,j_1} : \dots : p_{n,j_n}] \mid 1 \leq j_i \leq a_i + 1, i = 1, \dots, n \right\}.$$

We define the forms $G_i = \prod_{j=1}^{a_i+1} (X_i - p_{i,j}X_0)$, for $i = 1, \dots, n$. Since $\sum_{j=1}^{a_i+1} p_{i,j} = 0$, the G_i 's are in M^\perp and they generate the ideal of \mathbb{X} , which, by the Apolarity Lemma, gives a minimal real Waring decomposition of M .

The family of decompositions shown above can be parametrized as follows. Each decomposition is in bijection with n binary forms of degree a_i (because the last zero is determined by the others) whose roots are all real and distinct. Each binary form of degree a_i is in the projective space $\mathbb{P}(\mathbb{R}[X_0, X_i]_{a_i})$ and sits in the complement of the discriminant. Furthermore, each of these binary forms is in the connected component consisting of binary forms whose roots are all real, i.e. hyperbolic binary forms. Thus, the n -fold product of the connected components consisting of hyperbolic binary forms of degrees a_i gives the desired parametrization.

We give a result on the number of real solutions of some family of complete intersections, which has a similar flavour of the Descartes' rule of signs in the context of systems of polynomial equations.

Theorem 3.4. *Let $2 \leq a_0 \leq \dots \leq a_n$. For $i = 1, \dots, n$, let $F_i \in \mathbb{R}[X_1, \dots, X_n]$ be a polynomial of degree at most $a_i - a_0$. Then the system of polynomial equations defined by*

$$(2) \quad \begin{aligned} G_1 = X_1^{a_1+1} + F_1 &= 0, \\ &\vdots \\ G_n = X_n^{a_n+1} + F_n &= 0, \end{aligned}$$

does not have $\prod_{i=1}^n (a_i + 1)$ real distinct solutions.

Proof. We give a proof by contradiction, assuming that the number of real distinct solutions is $\prod_{i=1}^n (a_i + 1)$. Let $I = (G_1, \dots, G_n) \subseteq \mathbb{R}[X_1, \dots, X_n]$ be the ideal generated by G_1, \dots, G_n . We consider the \mathbb{R} -algebra $A = \mathbb{R}[x_1, \dots, x_n]/I$ and the bilinear form

$$B : A \otimes A \rightarrow \mathbb{R}, (H, K) \mapsto \text{Tr}_{A/\mathbb{R}}(m_{H \cdot K}).$$

Since the system has $\prod_{i=1}^n (a_i + 1)$ real distinct solutions, B is positive definite by Proposition 2.6. The residue classes of the monomials $X_1^{\alpha_1} \dots X_n^{\alpha_n}$ with $0 \leq \alpha_i \leq a_i$ form a basis of A as a vector space over \mathbb{R} . We want to show that the representing matrix M of the \mathbb{R} -linear map

$$\varphi : A \rightarrow A, H \mapsto X_1^2 \cdot H$$

with respect to this basis has only zeros on the diagonal. This would imply $B(X_1, X_1) = \text{Tr}_{A/\mathbb{R}}(m_{X_1^2}) = 0$, which in turn would imply that B is not positive definite.

For $0 \leq \alpha_i \leq a_i$ we have $\varphi(X_1^{\alpha_1} \dots X_n^{\alpha_n}) = X_1^{\alpha_1+2} \cdot X_2^{\alpha_2} \dots X_n^{\alpha_n}$. If $\alpha_1 + 2 \leq a_1$, then the column of M corresponding to the basis element $X_1^{\alpha_1} \dots X_n^{\alpha_n}$ has its only nonzero entry at the row corresponding to the basis element $X_1^{\alpha_1+2} \cdot X_2^{\alpha_2} \dots X_n^{\alpha_n}$. If $\alpha_1 + 2 > a_1$, then $\varphi(X_1^{\alpha_1} \dots X_n^{\alpha_n}) = -F_1 \cdot X_1^{\alpha_1+1-a_1} \cdot X_2^{\alpha_2} \dots X_n^{\alpha_n}$. It follows from our assumptions on the degrees of the F_i 's, that the element $\varphi(X_1^{\alpha_1} \dots X_n^{\alpha_n})$ is in the span of all basis elements corresponding to monomials of degree smaller than $\sum_{i=1}^n \alpha_i$. In both cases, the corresponding diagonal entry of M is zero. This concludes the proof. \square

We now give a characterization for those monomials whose real and complex ranks coincide.

Theorem 3.5. *Let $M = x_0^{a_0} \dots x_n^{a_n}$ be a degree d monomial with $0 < a_0 \leq \dots \leq a_n$. Then*

$$\text{rk}_{\mathbb{R}}(M) = \text{rk}_{\mathbb{C}}(M)$$

if and only if $a_0 = 1$.

Proof. If $a_0 = 1$, Corollary 3.2 proves the statement. Suppose that $\text{rk}_{\mathbb{R}}(M) = \text{rk}_{\mathbb{C}}(M)$ and let \mathbb{X} be a minimal set of real points apolar to M . Assume by contradiction that $a_0 \geq 2$. By Theorem 2.3, we know that \mathbb{X} is a complete intersection and we may dehomogenize by $X_0 = 1$. The set \mathbb{X} gives the solutions to a system of polynomial equations of the form (2) in Theorem 3.4. This is a contradiction and this concludes the proof. \square

Finally, we show that the upper bound in Proposition 3.1 is not always sharp.

Proposition 3.6. *Let $M = x_0^2 \dots x_n^2$. Then $\text{rk}_{\mathbb{R}}(M) \leq (3^{n+1} - 1)/2$.*

Proof. We explicitly give an apolar set of points for M as follows. For any $i = 0, \dots, n$, let us consider the set

$$\mathbb{X}_i = \left\{ [p_0 : \dots : p_{i-1} : 1 : p_{i+1} : \dots : p_n] \in \mathbb{P}^n \mid p_i \in \{0, \pm 1\} \right\}.$$

We can easily determine the cardinality of $\mathbb{X} = \bigcup_{i=0}^n \mathbb{X}_i$. From all $(n+1)$ -tuples (p_0, \dots, p_n) with $p_i = 0, \pm 1$, we need to discard $(0, \dots, 0)$, since it does not correspond to any point in the projective space. We are double counting, since (p_0, \dots, p_n) and $(-p_0, \dots, -p_n)$ define the same point in the projective space. Thus, $|\mathbb{X}| = (3^{n+1} - 1)/2$. For each $P \in \mathbb{X}$, let L_P denote the corresponding linear form $p_0 x_0 + \dots + p_n x_n$ and $n(P)$ the number of entries different from zero. For each $i = 1, \dots, n+1$, we set

$$R_i = \sum_{\substack{P \in \mathbb{X} \\ n(P)=i}} L_P^{2n+2}.$$

By direct computation, we obtain

$$\frac{(2n+2)!}{2} x_0^2 \dots x_n^2 = \sum_{i=1}^{n+1} (-2)^{n+1-i} R_i.$$

Thus, \mathbb{X} is apolar to M and this concludes the proof. \square

Example 3.7. For $n = 1$, we have the following real decomposition of $M = x_0^2 x_1^2$:

$$12x_0^2 x_1^2 = R_2 - 2R_1 = (x_0 + x_1)^4 + (x_0 - x_1)^4 - 2(x_0^4 + x_1^4).$$

For $n = 2$, we have $\text{rk}_{\mathbb{C}}(x_0^2 x_1^2 x_2^2) = 9$ and $10 \leq \text{rk}_{\mathbb{R}}(x_0^2 x_1^2 x_2^2) \leq 13$:

$$\begin{aligned} 360x_0^2 x_1^2 x_2^2 &= R_3 - 2R_2 + 4R_1 = \\ &= (x_0 + x_1 + x_2)^6 + (x_0 + x_1 - x_2)^6 + (x_0 - x_1 + x_2)^6 + (x_0 - x_1 - x_2)^6 + \\ &\quad - 2[(x_0 + x_1)^6 + (x_0 - x_1)^6 + (x_0 + x_2)^6 + (x_0 - x_2)^6 + (x_1 + x_2)^6 + (x_1 - x_2)^6] + \\ &\quad + 4(x_0^6 + x_1^6 + x_2^6). \end{aligned}$$

In [12, Example 6.7], it is proved that $\text{rk}_{\mathbb{R}}(x_0^2 x_1^2 x_2^2) > 10$.

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